Indian Statistical Institute, Bangalore

B.Math (Hons.) I Year, Second Semester

Semestral Examination

Time: 3 hours | 30 April 2012 | Instructor: C.R.E.Raja

Analysis II | Maximum marks: 50

Section I: Answer any four and each question is worth 6 marks

1. Let E be a subset of a metric space X. Then show that $x \in X$ is a limit point of E if and only if there is a sequence (x_n) in E such that $x_n \to x$ and $x_n \neq x$.

2. Let $f\colon [a,b] \to \mathbb{R}$ be a monotonically increasing function. Prove that $f \in \mathcal{R}[a,b]$.

3. Let $f \in \mathcal{R}[a,b]$ and $\epsilon > 0$. Prove that there is a continuous real-valued function g on $[a,b]$ such that $\int_a^b |f-g| < \epsilon$.

4. Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function. Suppose $f \in \mathcal{R}[c,b]$ for all $c \in (a,b)$. Prove that $f \in \mathcal{R}[a,b]$ and $\int_a^b f = \lim_{c\to a} \int_c^b f$.

5. Let E be an open subset of $\mathbb{R}^n$ and $f\colon E \to \mathbb{R}^m$ be a differentiable function. Prove that partial derivatives exist on E and obtain $f'(x)$ in terms of partial derivatives.

6. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \frac{x^3}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Show that $\lim_{t\to 0} \frac{f(tu)}{t}$ exists for any $u \in \mathbb{R}^2$ and find $D_i f$ for $i = 1,2$. Is f differentiable at $(0,0)$?

Section II: Answer any two and each question is worth 13 marks

1. (a) Prove that compact subsets of metric spaces are closed.

   (b) Let A and B be closed sets in $\mathbb{R}$.

   (i) Prove that $A \times B = \{(a,b) \in \mathbb{R}^2 \mid a \in A, \quad b \in B\}$ is a closed subset of $\mathbb{R}^2$.

   (ii) Further if A is bounded, prove that $A + B = \{a+b \in \mathbb{R} \mid a \in A, \quad b \in B\}$ is closed in $\mathbb{R}$.

2. (a) Let $f \in \mathcal{R}[a,b]$ and $F\colon [a,b] \to \mathbb{R}$ be given by $F(a) = 0$ and $F(t) = \int_a^t f$ for $t \in (a,b]$.

   (i) Prove that F is continuous on $[a,b]$.

   (ii) Further if f is continuous, then prove that F is differentiable and $F' = f$.

   (b) If $f \in \mathcal{R}[a,b]$, prove that $\lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f(a + k\frac{(b-a)}{n}) = \int_a^b f(x)dx$.

3. Let E be an open set in $\mathbb{R}^n$ and $f\colon E \to \mathbb{R}^m$ be a function with partial derivatives.

   (a) If partial derivatives of f are bounded, prove that f is continuous on E.

   (b) If partial derivatives of f are continuous, prove that f is differentiable on E.